Mathematical Thinking Framework

1. All learners can think mathematically.

By the time they reach school, all learners have demonstrated a significant range of innate ways of thinking that can be harnessed in the classroom to develop mathematical thinking. These skills are not specific to learning mathematics – they are part of everyday life. Being aware of these human powers enables teachers to provide tasks and activities that provide opportunities to become aware of and hone their mathematical thinking. Mathematical thinking is not the preserve of a specific group of learners, and activities that provoke mathematical thinking should be an essential part of the everyday classroom experience for all learners.

When learners categorise in everyday life (by comparing and contrasting features), they are thinking mathematically.
When learners imagine what might happen (based on their existing mathematical knowledge), they are thinking mathematically.
When learners find themselves wondering if something that has happened in a particular case might be an illustration of a more general idea, they are thinking mathematically.
When learners express a conjecture, they are thinking mathematically.
When learners find ways of demonstrating that something must be true, they are thinking mathematically.

How would you answer these questions?

- As you plan activities, and the questions and prompts that you will use alongside the activities, consider the following questions:
  If you want learners to think mathematically by:
  - **classifying**, then what opportunities will learners have to classify?
  - **sorting**, then what opportunities will learners have to sort?
  - **comparing and contrasting**, then what opportunities will learners have to compare / contrast? What questions will you ask learners to draw their attention to distinctive features?
  - **imagining**, then what situations might you create for learners so that imagining what might happen would be part of their response?
  - **conjecturing**, then what questions might you ask that prompt learners to conjecture? What support / scaffolding might learners need in order to articulate conjectures?
  - **generalising**, then how could the activities that you provide prompt them to see beyond the immediate situation, in order to see the universal applicability of what they are learning?
  - **proving**, then what can you do to support learners to be clear both about what it is that they are trying to prove, and how to express each stage of their thought so that it is logically coherent?

2. An important skill of the mathematics teacher is being able to recognise and plan for the mathematical thinking learners will need.

Mathematical thinking is not something that can be seen, yet it is of vital importance. Teachers of mathematics learn to recognise the outward manifestations of learners' mathematical thinking and actively consider how to set tasks that provoke mathematical thinking.

Mathematical thinking is about the nature of thinking, not about incorrect or correct outputs from learners. It is about the journey more than it is about the destination. Therefore, in developing learners' mathematical thinking, consideration is needed on the part of the teacher of the learning journey itself, and how learners can become better at taking these journeys unaided. The tasks that the teacher selects, and the ways that they work with learners on these tasks, need to be considered from the perspective of how they impact upon the learning journey for the learners. This is particularly important when the mathematics itself is so intuitive to the teacher, that their own thinking will be markedly different from that of the learner.

How would you answer these questions?

- If a learner 'gets the answer right' in your lesson, does this necessarily mean that they have been thinking mathematically? How could you find out?
- What is the difference between a correct explanation that has been recalled from long-term memory, and an equally correct explanation that is the immediate expression of fresh reasoning? What type of thinking does each reveal? How can you tell which one a learner is doing?
- What is the thinking that you intend the learners to engage in? How is this thinking supported by the tasks that you have chosen? What are the questions and prompts (or other actions) that you could use to provoke the intended thinking?

3. Mathematical thinking requires effort.

Mathematicians get stuck when working on mathematics; this is an expected part of the process. Teachers can sometimes feel the need to simplify cognitive tasks for learners, so that they achieve 'success'. If learners always have tasks broken down into small, clearly defined steps, then this limits the types of mathematical thinking required on the part of the learner. This loss of opportunity to grapple with mathematical ideas can lead to a lack of resilience and a learned helplessness. Any teacher interventions should aim to empower the learner to be able to 'think for themselves'.

How would you answer these questions?

- What are the 'difficulty points' in the mathematics you are teaching?
- When considering the tasks that you intend to use in your lesson:
  What are the steps / connections / underlying ideas that are obvious to you, but might not be obvious to the learners? How can you prompt learners to see what you see? Will you just tell them, or are their other options available to you that will support the learner to think mathematically?
- How do you manage expectations in your classroom so that learners recognise that getting stuck is part of the process of thinking mathematically? Do your learners have opportunities to get stuck?
- Do the learners see you genuinely working on mathematical tasks? Do you invite them to participate? Do you model the process of being stuck? Do they see you demonstrating resilience?

4. Thinking mathematically often involves reshaping previously held ideas.

As learners deepen their understanding, this may involve re-evaluating, re-forming, connecting or rejecting previous ideas. It is important that learners are allowed to experience the cognitive conflict involved in order to achieve this. This may involve recognising that:

- Ideas they had previously held were incorrect (e.g. thinking that division must be done before multiplication)
- Ideas they had thought were ‘always true’ are in fact only ‘sometimes true’ (e.g. thinking that subtracting leads to a smaller result)
- Ideas they thought were completely separate are actually linked (e.g. factors and square numbers)
- Areas of mathematics that they thought were distinct can actually be considered as having the same structures (e.g. ratio and percentages)
- Categories that they thought were distinct, do in fact overlap (e.g. rectangles and squares)
- Symbols that they use in mathematics might have meanings different from the meanings that they had previously given them (e.g. the equals symbol represents equality, not ‘the answer is’)
- Calculations that they thought meaningless (e.g. 2 - 7) actually have meaning
- ‘Processes’ can also be considered as concepts in their own right (e.g. the process of subtracting 5 from 0, yields the value -5, which can be considered a concept in its own right)

How would you answer these questions?

- What ideas might learners encounter in your lesson that could (and indeed should) conflict with their previously held ideas? How will their thinking be reshaped / built upon?

- How will you frame the way learners engage with the necessary reshaping of their understanding? Will you just ‘tell’ them? And if their previous understanding is strongly held, will they believe you? Will you offer them examples that allow them to experience a (possibly uncomfortable) shift in perception? How will you guide them through this?

- What might learners’ responses be? What questions might suddenly emerge for them? What will they need to do in order to reshape their understanding? How will the activities that you plan support them in this? Consider:
  - The sentences you will say
  - The sentences they will need to say
  - The questions you will ask
  - The questions they will need to ask
  - The written questions they work on
  - The representations they will use
  - The mathematical problems they will work on

- What if learners resist the cognitive conflict?
  What prompts / questions / scaffolds do you have ready to (a) find out exactly what it is in their thinking that is causing the reluctance to reshape their thinking? and (b) what further examples / representations / language structures do you have ready?

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5. Mathematicians develop ‘habits of mind’ to think mathematically.

Mathematicians get stuck when doing mathematics. This is an expected part of the process. Getting ‘unstuck’ requires resilience and learners become more resilient when they have a greater range of thinking strategies upon which to draw. This mental toolkit supports them to think like mathematicians. The tools are the mathematician’s ‘habits of mind’, and learners need to both develop and become aware of these. One way in which teachers can support learners in developing their ‘habits of mind’ is by modelling aloud their thought processes and approaches when working on mathematics.

Mathematicians play with (or manipulate) ideas. This takes place as they begin to engage with a new problem, or a new concept. During the ‘manipulating’ stage, they find different ways to interact with the mathematics. They might:

- try to draw, make, or otherwise represent the problem / concept.
- try to rephrase the situation.
- try particular numbers, or particular shapes, to see what happens.
- continue to try particular cases, working systematically.
- vary or change some aspects, but not others.
- make comparisons.
- try to put things into groups.
- try different ‘ways in’, hoping that one of these will work.
- repeat certain steps, in the hope that eventually this might lead to an insight developing.

As they ‘manipulate’ the mathematics, they develop insights. These insights allow them to get a sense of the underlying mathematical structures. As this continues, they begin to articulate this thinking by communicating it verbally, or in written form, or with concrete, pictorial or more abstract written representations.

How would you answer these questions?

- What are the main mathematical ideas in your lesson?
  What is unexpected about these ideas? What is challenging about these ideas? In what ways do these ideas link to other parts of mathematics?
  How will the tasks you have planned invite learners to play with the ideas so that they engage with the essential ideas and see the connections?

- Consider the problem-solving tasks in your lesson.
  What might learners need to do to find a way in?
  What do you need to make available in your lesson to support learners to find ways in?

- Once learners have found a way in, would working systematically help?
  What choices do they need to make for themselves in order to plan what ‘system’ to use? Are there alternative ‘systems’ of working? What might be the advantages of different approaches?

- What thinking will you model aloud? What will you say? When?

- What opportunities will you provide for learners to reflect on the types of thinking they have done and the choices they have made, so that they can deploy these actions again in future?